Nonlinear Fibre Optics

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Content

I. **Generalities on nonlinearities in silica**
   - Definition, phenomenology and classification
   - Material response to an optical wave
   - Dispersion, dispersion length, loss
   - Nonlinear effective length and area

II. **Optical Kerr effect**
   - Overview of third order nonlinear parametric processes
   - Self-phase
   - Nonlinear pulse propagation
   - Solitons
   - Modulation instability
   - Solving the NLSE
   - Multiwavelength effects: cross phase modulation and four-wave mixing
   - Parametric amplification
   - Supercontinuum generation

III. **Stimulated scatterings**
   - Phenomenology, spontaneous scatterings
   - The case of elastic Rayleigh scattering
   - Spectral characteristics, energy-momentum conservation
   - Stimulated Raman scattering
   - Stimulated Brillouin scattering
   - Amplification, laser

IV. **Applications using stimulated scatterings**
   - Power probing using stimulated Brillouin scattering
   - Distributed Raman fibre sensing
   - Distributed Brillouin fibre sensing
   - Slow & fast light
   - Dynamic Brillouin gratings
   - Optical storage

Preamble

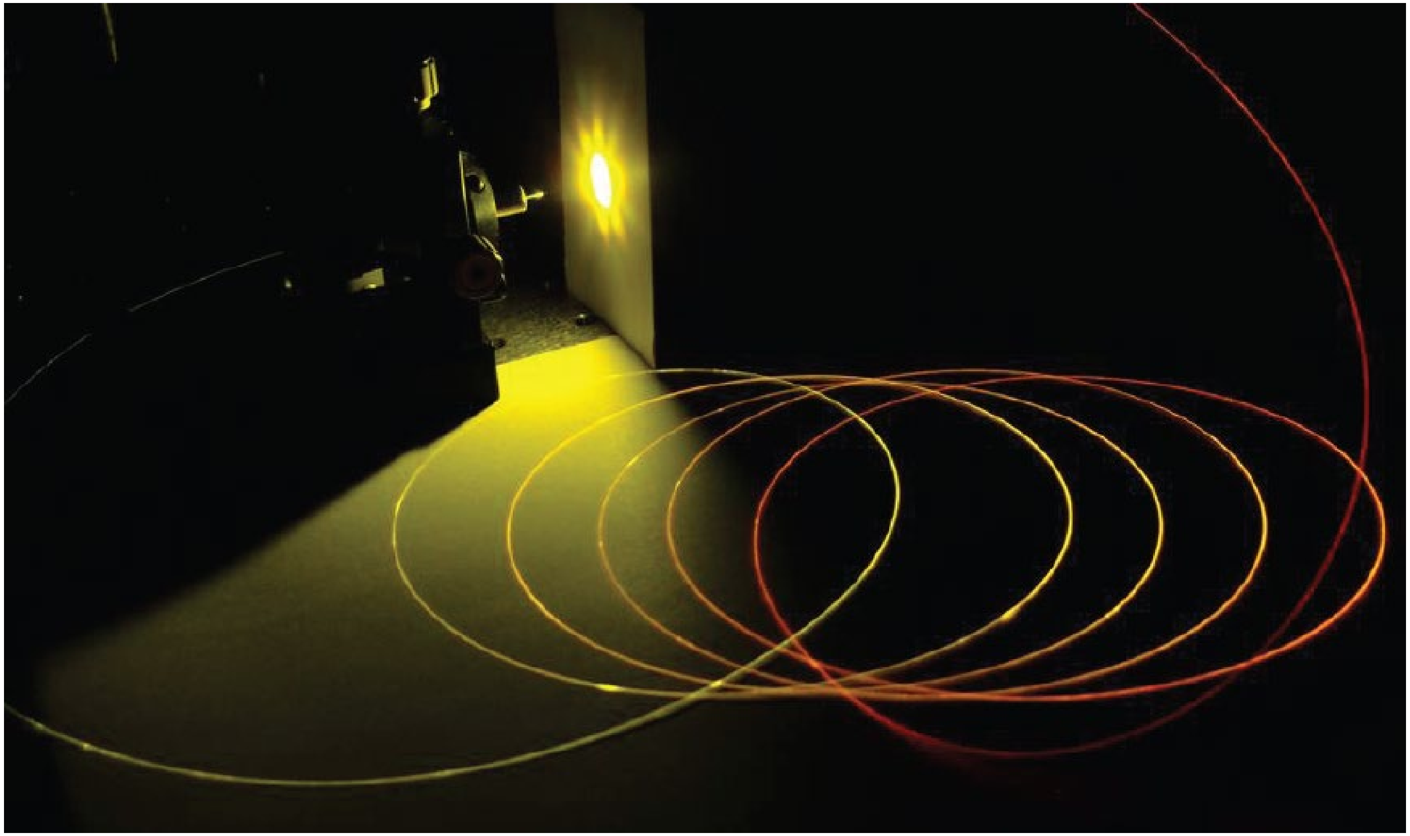
- **In ordinary observation conditions**: the response of the medium to the presence of the lightwave can be accurately considered as linear.

Consequences:

1. The optical properties (n, α) of the medium are **independent** of the light **intensity**.
2. The **principle of superposition** is relevant in all situations.
3. The spectral content of light cannot be **enriched**, i.e. the spectral band of the light signal at the output of the optical medium is equal to or lesser than the bandwidth of the input signal.

Preamble

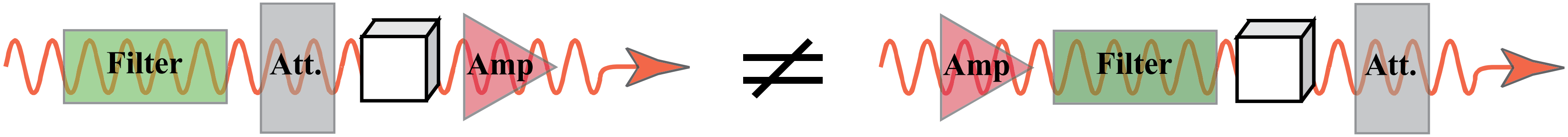
Nonlinear optical effects are striking phenomena that arise when an intense optical beam propagates through a medium



Preamble

- **The possibility to deliver strong intensities** using lasers made possible observations that can be only explained by a **non-linear** response of the medium, such as:

  1. A **change of the index of refraction** when the light becomes intense.
  2. A **change of the light spectrum** after propagation in a nonlinear medium.
  3. Numerous situations in which the **principle of superposition is violated**; for instance the control of a light beam by another light beam is possible ("**optical transistor**").



Preamble

Presently, nonlinear effects are mainly considered detrimental in transmission systems, HOWEVER

- Nonlinear effects are the only route to control light by light
- Not so far from reality, applications already in use include:
  - Robust long-haul transmission in optical networks
  - Wavelength conversion and data demultiplexing
  - Optical regeneration
  - All-optical switching
  - Distributed sensing
  - Supercontinuum sources (e.g. for medical applications)
- Essential in future high-capacity (all-optical) networking!!

Basics

Nonlinear medium ↔ The polarization field $\mathscr{P}$ is no longer proportional to the incident field $\mathscr{E}$

Nonlinear effects are **weak** ➔ Taylor's limited expansion of $\mathscr{P}$ over $\mathscr{E}$

$$\begin{aligned}\mathscr{P} &= a_1\,\mathscr{E} + \frac{1}{2}a_2\,\mathscr{E}^2 + \frac{1}{6}a_3\,\mathscr{E}^3 + \dots \\ &= \varepsilon_o\,\chi\,\mathscr{E} + 2\,d\,\mathscr{E}^2 + 4\,\chi^{(3)}\,\mathscr{E}^3 + \dots\end{aligned}$$

- Scalar approximation of the expansion.

- **Using real fields is required** (no complex amplitude!)

Basics

Wave equation obtained from Maxwell equations in a homogeneous dielectric medium:

$$\nabla^2 \mathcal{E} - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \mathcal{E} = \mu_o \frac{\partial^2}{\partial t^2} \mathcal{P}$$

Decomposition of the polarization into a *main linear contribution* and a *nonlinear correction term*:

$$\mathcal{P} = \varepsilon_o \, \chi \, \mathcal{E} + \mathcal{P}_{NL}$$

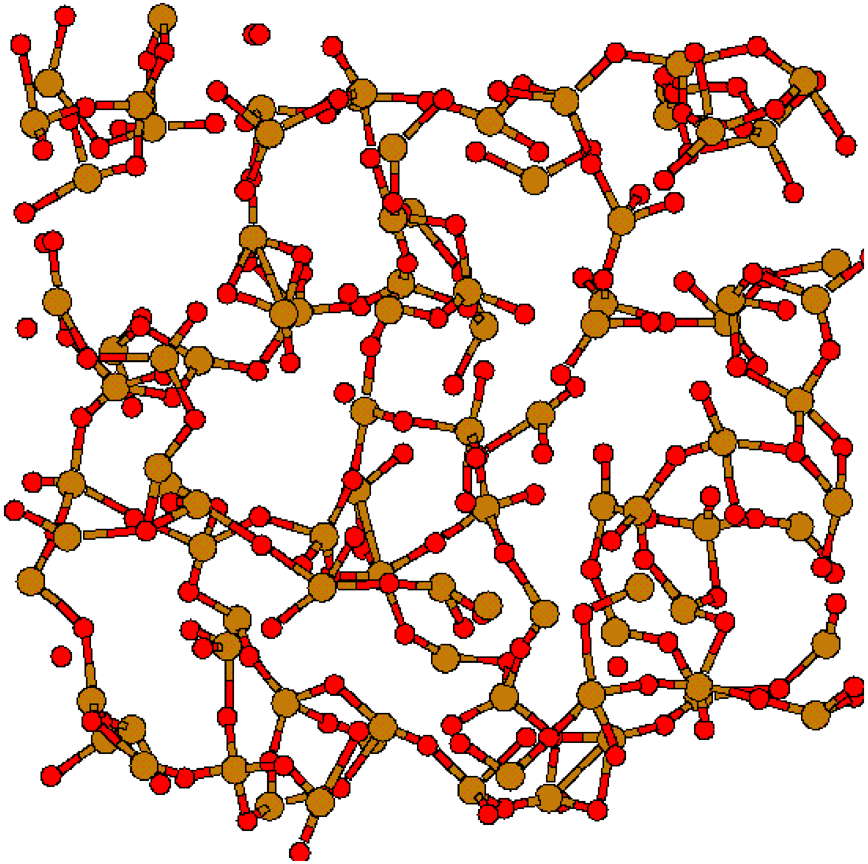
Wave equation:

$$\boxed{\nabla^2 \mathcal{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathcal{E} = \mu_o \frac{\partial^2}{\partial t^2} \mathcal{P}_{NL}}$$

Nonlinear term: *source* term in the wave equation *radiating* into a linear medium of index *n*.

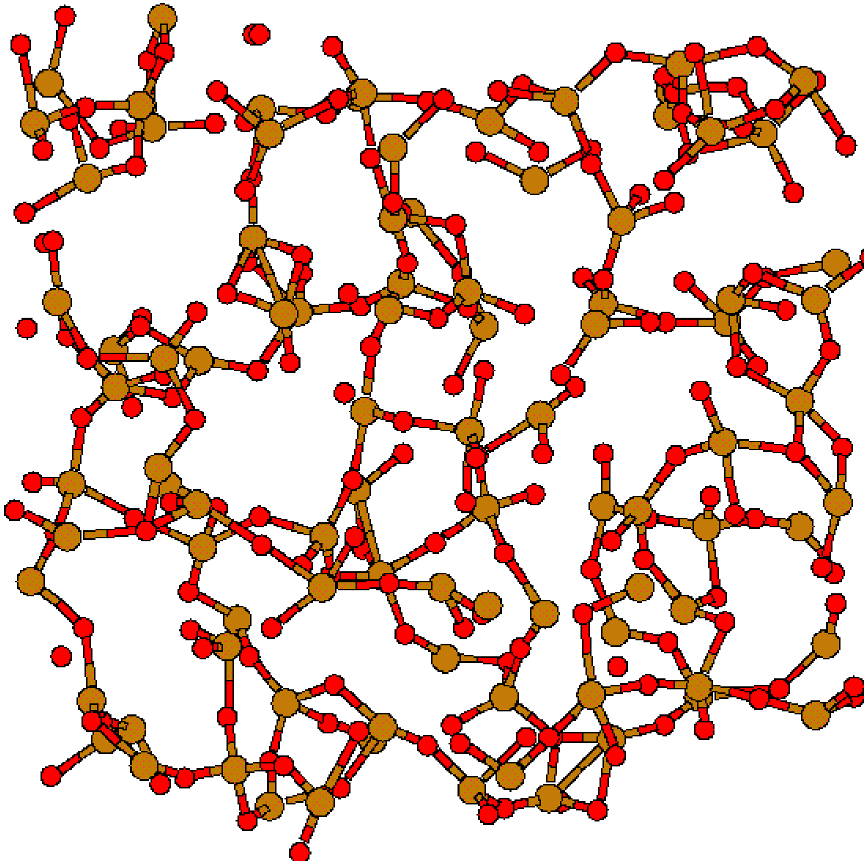
Since this term is *weak*, it is usually handled as a *perturbation*.

Nonlinearities in optical fibres



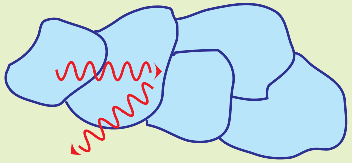
- Silica symmetry ⇒ 2nd-order non-linearities are negligible
- Third-order non-linear response
- Power densities ≥ 0.1 GW/m^2
  - Already reached with P~10mW in standard fibres
- Quasi-linear response

Nonlinearities in optical fibres



- Two types of nonlinear effects can be distinguished:
  - Inelastic scattering effects (Raman, Brillouin): the incident photons are annihilated to create a (normally) lower energy photon and a phonon (net energy loss)
  - Elastic effects (Kerr effects): the incident photons suffer phase and/or frequency shifts but overall there is no energy loss.

Optical effects in fibres

	Medium reaction	**Scatterings**
Linear effects	Medium polarization ↓ Index of refraction $n = \sqrt{\varepsilon}$ ↓ Dispersion $n(\nu)$ - Absorption $\alpha(\nu)$	**Elastic** Rayleigh scattering  **Spontaneous inelastic scatterings (phonon-photon)** Brillouin scattering - Raman scattering
Nonlinear effects	Medium polarization ↓ $\vec{\mathcal{P}} = \underline{\underline{\varepsilon}}(\vec{\mathcal{E}})\,\vec{\mathcal{E}}$ Index of refraction $n = n_o + n_2 I$ ↓ **Altered spectrum** Harmonic generation - Pulse compression - Solitons	**Stimulated inelastic scatterings** Scattered wave sustains the scattering wave ↓ Amplification of the scattered wave ↓ **Gain** Brillouin scattering - Raman scattering

Response of a linear optical medium

$$dE(z,\omega) = \underbrace{[-\frac{\alpha(z,\omega)}{2}+j\beta(z,\omega)]}_{H(z,\omega)}\, E(z,\omega)\, dz$$

Microscopic
transfer function

$$\Rightarrow \quad dE(z,t)=h(z,t)\otimes E(z,t)\,dz \quad \text{where} \quad h(z,t)=FT^{-1}\{H(z,\omega)\}$$

Impulse response

Since the field $E(z,t)$ is *real*, the impulse response $h(z,t)$ must be *real*, too.

➔ $H(z,\omega)$ is a *Hermitian* function: *real* part is *even*, *imaginary* part is *odd*.

➔ This suggests that pure *amplitude* effects (real part) and *phase* shifts are mutually dependent.

➔ $H(z,\omega)$ is *frequency-independent* only if $h(z,t)$ is equal to a *Dirac* response $\delta(t)$. In this case the medium response is **instantaneous** and the propagation is **lossless**. Only medium fulfilling this condition: **vacuum**!

Dispersion in optical fibres

In a longitudinally homogeneous medium, α and β are independent of z and the microscopic transfer function can be integrated:

$$E(\omega,0) \rightarrow \quad E(\omega,z)=E(\omega,0)\,\mathrm{e}^{-\frac{\alpha(w)}{2}z}\,\mathrm{e}^{j\beta(w)z}$$

In a dense medium, the response is non-instantaneous: the propagation constant $\beta(\omega)$ is *frequency-dependent* and creates different *phase shifts* for different spectral components of the signal.

$$\beta(\omega)=n_{eff}(\omega)\frac{\omega}{c_o}=\beta_o+\beta_1(\omega-\omega_o)+\frac{1}{2}\beta_2(\omega-\omega_o)^2+\frac{1}{6}\beta_3(\omega-\omega_o)^3+\frac{1}{24}\beta_4(\omega-\omega_o)^4+\cdots$$

$\beta_o=n(\omega_o)\dfrac{\omega_o}{c_o}$: Global phase shift for all signal components, *not observable* on the intensity waveform.

Dispersion in optical fibres

$$\beta(\omega)=n_{eff}(\omega)\frac{\omega}{c_o}=\beta_o+\beta_1(\omega-\omega_o)+\frac{1}{2}\beta_2(\omega-\omega_o)^2+\frac{1}{6}\beta_3(\omega-\omega_o)^3+\frac{1}{24}\beta_4(\omega-\omega_o)^4+\cdots$$

(Group velocity)$^{-1}$

$$\beta_1=\frac{1}{V_g}=\frac{n_g}{c_o}=\frac{n+\omega_o\dfrac{dn}{d\omega}}{c_o}$$

Group velocity dispersion (GVD)

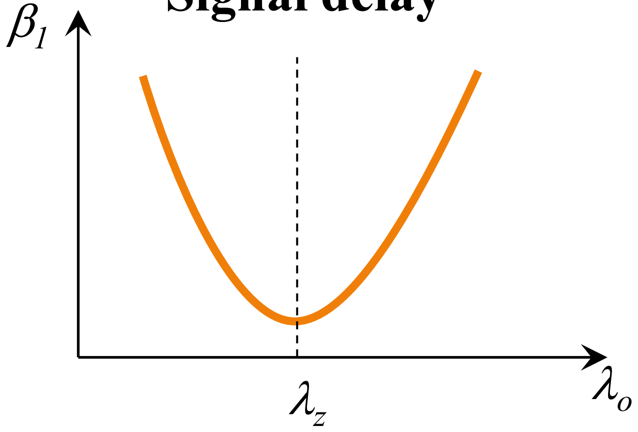
$$\beta_2=\frac{d^2\beta}{d\omega^2}=\frac{d}{d\omega}(\frac{1}{V_g})=\frac{2\dfrac{dn}{d\omega}+\omega_o\dfrac{d^2n}{d\omega^2}}{c_o}$$

[ps^2/km]

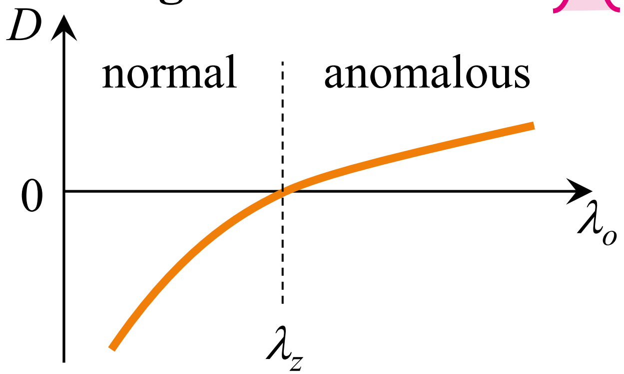
$$D=\frac{d\beta_1}{d\lambda_o}=-\frac{2\pi c_o}{\lambda_o^2}\beta_2$$

[ps/nm·km]

Signal delay



Signal distortion



Dispersion length

(Pulse of width τ)

$$L_{GVD}=\frac{\tau^2}{|\beta_2|}=\frac{2\pi\tau^2 c_o}{\lambda_o^2|D|}$$

Slowly varying envelope approximation

Assumptions:

- The propagation of the light wave is unidirectional in the *z* direction $\xrightarrow{z}$
- The determination of the transversal distribution of the field $F(x,y)$ is solved and found separately.
- The carrier at angular frequency ω_o is modulated by an envelope $A(z,t)$ at a speed much slower than the carrier frequency.

The field can be expressed as: $E(x,y,z,t) = A(z,t)F(x,y)\mathrm{e}^{j(\omega_o t-\beta_o z)}$ with $\beta_o = \beta(\omega_o)$

Let consider the wave equation without perturbation:

$$\nabla^2 E(x,y,z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(x,y,z,t) = 0$$

which can be rewritten with the set of assumptions:

$$\frac{\partial^2}{\partial z^2}A(z,t)\mathrm{e}^{j(\omega_o t-\beta_o z)} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A(z,t)\mathrm{e}^{j(\omega_o t-\beta_o z)} = 0$$

Slowly varying envelope approximation

Variation of $A(z)$ over the distance $\Delta z=\lambda$: $|\Delta A| \ll |A|$

$$\Delta A = \left(\frac{\partial A}{\partial z}\right)\Delta z = \left(\frac{\partial A}{\partial z}\right)\lambda \Rightarrow \left|\frac{\partial A}{\partial z}\right| = \left|\frac{\Delta A}{\lambda}\right| << \left|\frac{A}{\lambda}\right| < |kA| \Rightarrow \left|\frac{\partial A}{\partial z}\right| << |kA|$$

Then :

$$\left|\frac{\partial^2 A}{\partial z^2}\right| = \frac{\left|\frac{\partial A}{\partial z_1} - \frac{\partial A}{\partial z_2}\right|}{\Delta z} < \frac{\left|\frac{\partial A}{\partial z_1}\right| + \left|\frac{\partial A}{\partial z_2}\right|}{\Delta z} \simeq \frac{2\left|\frac{\partial A}{\partial z}\right|}{\lambda} << k^2|A| \Rightarrow \left|\frac{\partial^2 A}{\partial z^2}\right| < k\left|\frac{\partial A}{\partial z}\right| << k^2|A|$$

Similarly with $\Delta t=T$: $\left|\frac{\partial^2 A}{\partial t^2}\right| < \omega_o\left|\frac{\partial A}{\partial z}\right| << \omega_o^2|A|$

The wave equation can be developed:

$$\cancel{\frac{\partial^2 A(z,t)}{\partial z^2}}e^{j(\omega_o t-\beta_o z)} - j2\beta_o\frac{\partial A(z,t)}{\partial z}e^{j(\omega_o t-\beta_o z)} - \beta_o^2 A(z,t)e^{j(\omega_o t-\beta_o z)}$$

$$-\frac{1}{c^2}\cancel{\frac{\partial^2 A(z,t)}{\partial t^2}}e^{j(\omega_o t-\beta_o z)} - j\frac{1}{c^2}2\omega_o\frac{\partial A(z,t)}{\partial t}e^{j(\omega_o t-\beta_o z)} + \frac{\omega_o^2}{c^2}A(z,t)e^{j(\omega_o t-\beta_o z)} = 0$$

➔ $-j2\beta_o\frac{\partial A(z,t)}{\partial z} - \beta_o^2 A(z,t) - j2\frac{\beta^2}{\omega_o}\frac{\partial A(z,t)}{\partial t} + \beta^2 A(z,t) = 0$ using $\beta = n(\omega)k_o = n(\omega)\frac{\omega_o}{c_o} = \frac{\omega_o}{c}$

Slowly varying envelope approximation

$$-j2\beta_o\frac{\partial A(z,t)}{\partial z}-\beta_o^2A(z,t)-j2\frac{\beta^2}{\omega_o}\frac{\partial A(z,t)}{\partial t}+\beta^2A(z,t)=0$$

Fourier Transform: $$-j2\beta_o\frac{\partial \tilde{A}(z,\omega)}{\partial z}-\beta_o^2\tilde{A}(z,\omega)+\cancel{2\frac{\beta(\omega)^2}{\omega_o}(\omega-\omega_o)\tilde{A}(z,\omega)}+\beta(\omega)^2\tilde{A}(z,\omega)=0$$

$$\beta(\omega)=n_{eff}(\omega)\frac{\omega}{c_o}=\beta_o+\beta_1(\omega-\omega_o)+\frac{1}{2}\beta_2(\omega-\omega_o)^2+\cdots$$ ➔ $$\beta^2\simeq\beta_o^2+2\beta_o\beta_1(\omega-\omega_o)+\beta_o\beta_2(\omega-\omega_o)^2$$

(assuming $\beta_o\gg\beta_1(\omega-\omega_o)+\frac{1}{2}\beta_2(\omega-\omega_o)^2+\cdots$)

$$-j2\cancel{\beta_o}\frac{\partial \tilde{A}(z,\omega)}{\partial z}\cancel{-\beta_o^2\tilde{A}(z,\omega)}\cancel{+\beta_o^2\tilde{A}(z,\omega)}+2\cancel{\beta_o}\beta_1(\omega-\omega_o)\tilde{A}(z,\omega)+\cancel{\beta_o}\beta_2(\omega-\omega_o)^2\tilde{A}(z,\omega)=0$$

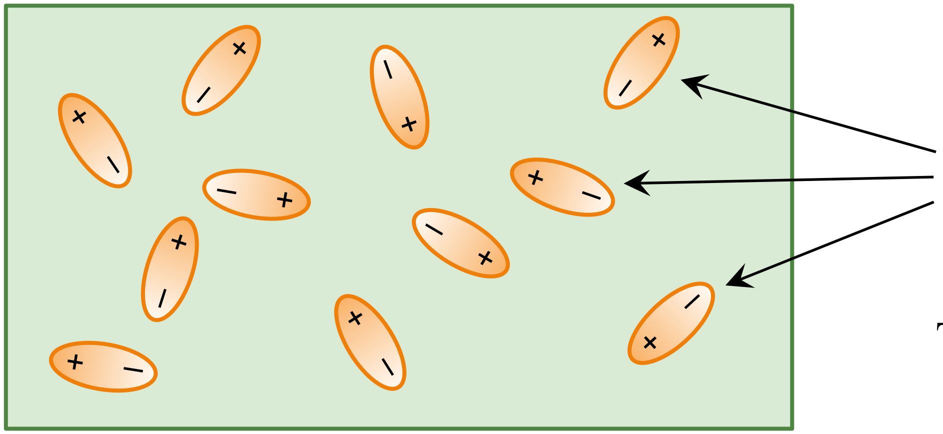
Inverse Fourier Transform: $$\frac{\partial A(z,t)}{\partial z}+\beta_1\frac{\partial A(z,t)}{\partial t}-\frac{j}{2}\beta_2\frac{\partial^2A(z,t)}{\partial t^2}=0$$

Perturbation terms can be added on the right hand side, e.g. linear loss:

$$\frac{\partial A(z,t)}{\partial z}+\beta_1\frac{\partial A(z,t)}{\partial t}-\frac{j}{2}\beta_2\frac{\partial^2A(z,t)}{\partial t^2}=-\frac{\alpha}{2}A(z,t)$$

Effect of electric field on dielectric material

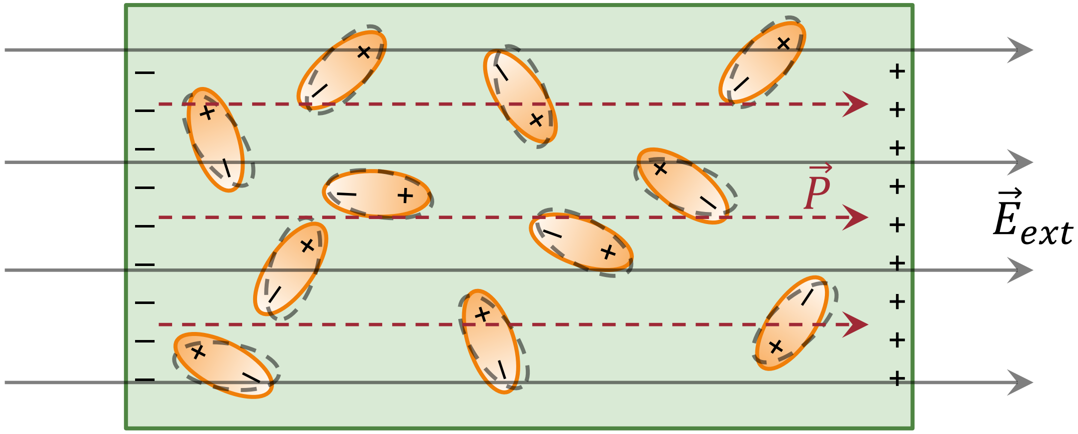
$\vec{E}_{ext} = 0$



Small dipoles (polar molecules)
Randomly oriented

The internal field averages to zero

$\vec{E}_{ext} \neq 0$



Small dipoles reorient
Average alignment of molecules

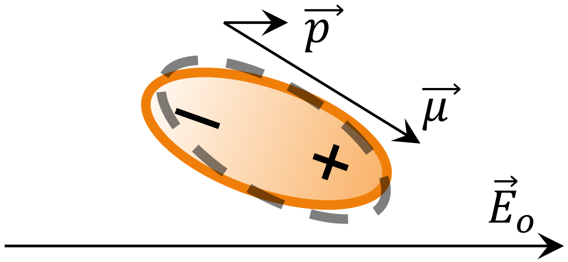
Surface charges
on dielectric boundaries

Internal polarization field $\vec{P}$

$$\boxed{\vec{P} = \varepsilon_o \chi \vec{E}_{ext}}$$

Microscopic reaction on electric field

In presence of an electric field the total dipole of a polar molecule is:



$$\vec{p}_{mol} = \vec{\mu} + \vec{p} = \vec{\mu} + \alpha\vec{E}_o$$

Intrinsic dipole

Induced dipole

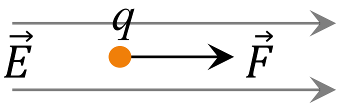
For a large number of randomly oriented molecules like in a dielectric dense medium the intrinsic dipoles mutually cancel while the induced dipoles are all aligned and sum up to give the macroscopic polarization field $\vec{P}$:

$$\vec{P} = \Sigma\,\vec{p}_{mol} = \underbrace{\cancel{\Sigma\vec{\mu}}}_{=0} + \Sigma\vec{p} = N\alpha\vec{E}_o$$

N: Molecule density [number/m^3]

α: Molecule polarisability $(\neq\chi!)$

A difficulty: the local electric field $\vec{E}_o$ is not equal to the total electric field, according to the general electric field definition (force $\vec{F}$ exerted on a probe charge q):

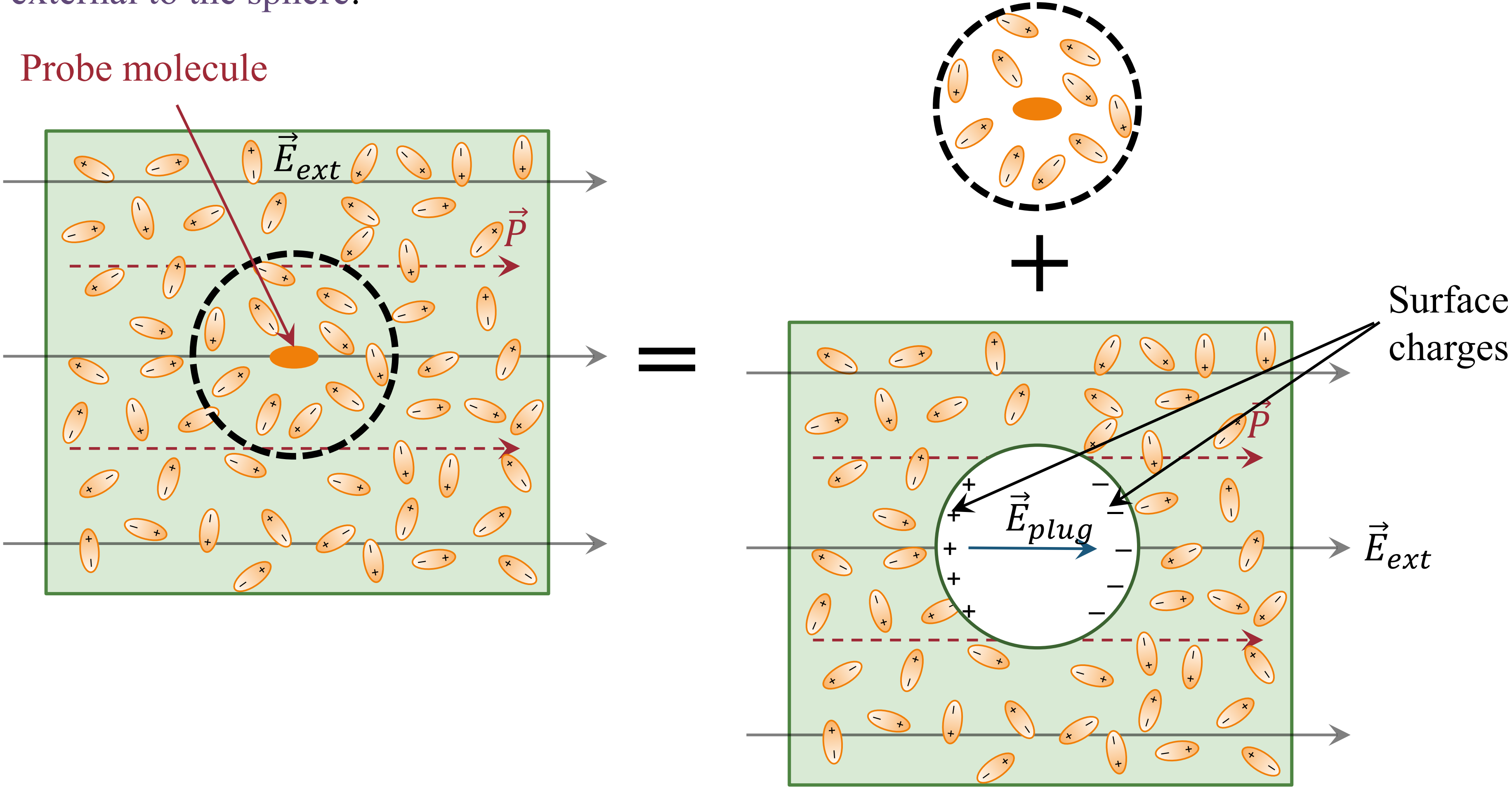


$$\vec{F} = q\vec{E}$$

The field of the probe charge is omitted !

Microscopic reaction on electric field

The local electric field is evaluated from the average applied external field by considering a **small sphere** centred on a given polar molecule and adding the fields internal and external to the sphere:

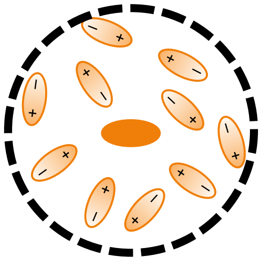


Microscopic reaction on electric field

The local electric field is evaluated from the average applied external field by considering a **small sphere** centred on a given polar molecule and adding the fields internal and external to the sphere:

Omitting the field from the probe molecule, the dipoles fields mutually cancel at the sphere centre, even if these dipoles are slighlty aligned by the external field.

$$\vec{E}_{sph} = 0$$

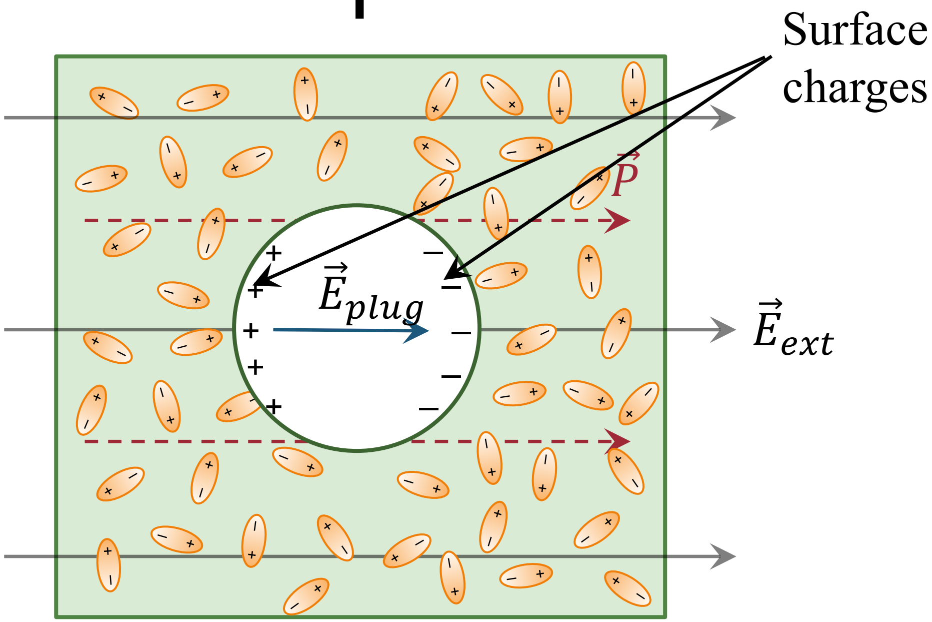


+

The field in the plug due to surface charges is equal to the polarization field $\vec{P}$, corrected by a 1/3 geometrical factor due to the sphericity:

$$\vec{E}_{plug} = \frac{1}{3}\frac{\vec{P}}{\varepsilon_o}$$

$$\vec{E}_o = \vec{E}_{ext} + \vec{E}_{sph} + \vec{E}_{plug} = \vec{E}_{ext} + \frac{1}{3}\frac{\vec{P}}{\varepsilon_o}$$



The Clausius-Mossotti relation

The expression for the local field $\vec{E}_o$ can be further developped:

- $\vec{E}_o = \vec{E}_{ext} + \frac{1}{3}\frac{\vec{P}}{\varepsilon_o} = \vec{E}_{ext}(1+\frac{\chi}{3})$ using $\vec{P} = \varepsilon_o \chi \vec{E}_{ext}$

But the local field $\vec{E}_o$ can also be expressed this way:

- $\vec{E}_o = \frac{\vec{P}}{N\alpha} = \frac{\varepsilon_o \chi}{N\alpha} \vec{E}_{ext}$

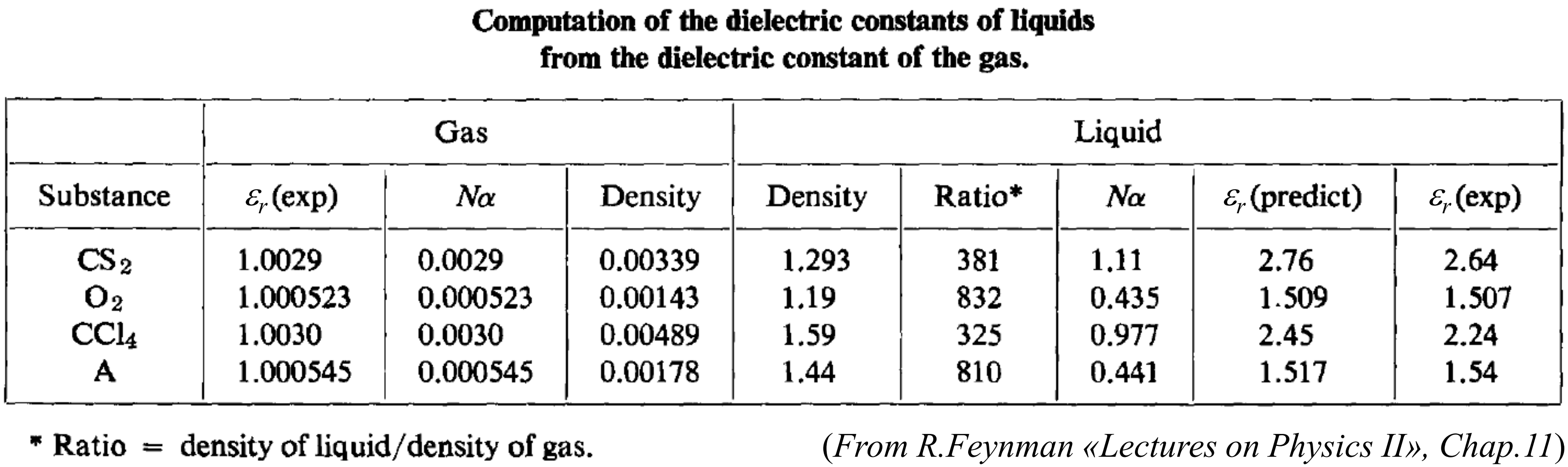
Equating the 2 expressions for the field $\vec{E}_o$: $\boxed{\frac{\chi}{\chi+3} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N}{3\varepsilon_o}\alpha}$ using $\varepsilon_r = 1 + \chi$

Clausius-Mossotti relation

This gives an essential relation between a microscopic quantity intrinsic to a molecule − the polarisability α − and a macroscopic quantity giving the global dielectric response of a material − the susceptibility χ.

For a tenuous material with $\chi << 1$ (low pressure gas), the dielectric response is proportional to the molecule density (≈ pressure): $\chi = N\alpha/\varepsilon_o$

The Clausius-Mossotti relation



Computation of the dielectric constants of liquids from the dielectric constant of the gas.

	Gas			Liquid				
Substance	ε_r(exp)	$N\alpha$	Density	Density	Ratio*	$N\alpha$	ε_r(predict)	ε_r(exp)
CS_2	1.0029	0.0029	0.00339	1.293	381	1.11	2.76	2.64
O_2	1.000523	0.000523	0.00143	1.19	832	0.435	1.509	1.507
CCl_4	1.0030	0.0030	0.00489	1.59	325	0.977	2.45	2.24
A	1.000545	0.000545	0.00178	1.44	810	0.441	1.517	1.54

* Ratio = density of liquid/density of gas.

(*From R.Feynman «Lectures on Physics II», Chap.11*)

In the optical regime, the Clausius-Mossotti relation can be expressed for the index of refraction $n = \sqrt{\varepsilon_r}$:

$$\boxed{\frac{n^2-1}{n^2+2} = \frac{N}{3\varepsilon_o}\alpha = \frac{N_A\,\rho\,\alpha}{3M}}$$

with

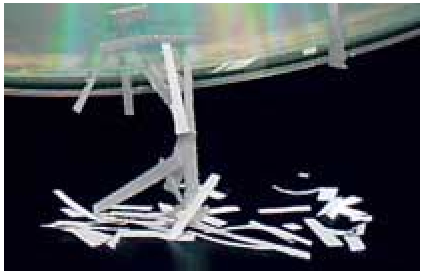
N_A : Avogadro's constant
ρ : Material density
M : Molar mass

The refractive index n depends on the material density ρ
➔ n will get higher when the medium is compressed:

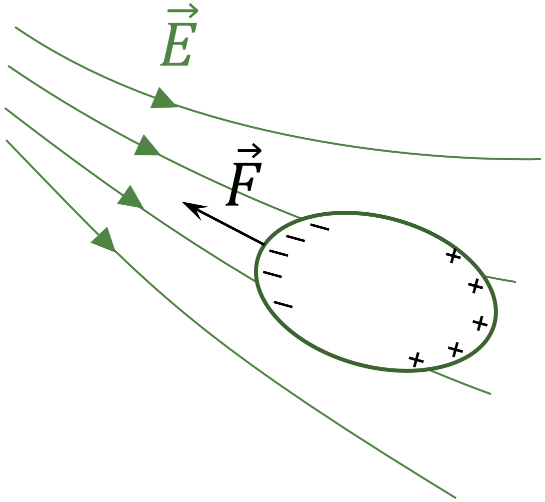
$$\boxed{\frac{dn}{d\rho} = \frac{(n^2-1)(n^2+2)}{6n\rho} > 0}$$

Elasto-optic effect

The electrostriction



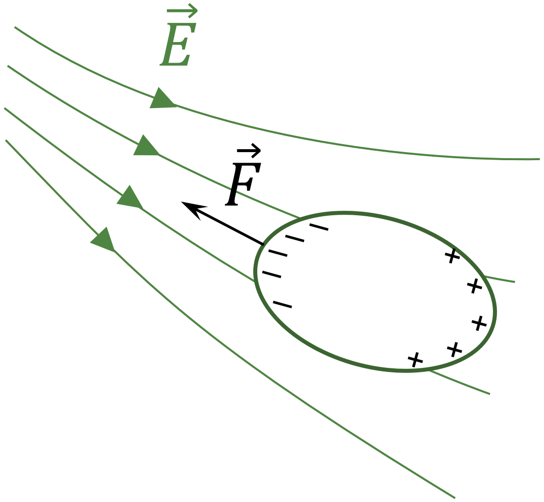
How uncharged bodies can be attracted by an electric field ?



A dielectric object in a non-uniform field feels a force toward regions of higher field strength.

$$\vec{F} \sim \underset{\substack{\nearrow\\ \text{Non-uniformity}\\ \text{strength}}}{\vec{\nabla}} \quad \underset{\substack{\uparrow\\ \text{Induced}\\ \text{charges}}}{\vec{E}} \quad \underset{\substack{\nwarrow\\ \text{Force}\\ \text{strength}}}{\vec{E}} \quad \sim \quad \vec{\nabla}\vec{E}^2$$

The electrostriction



The exact expression for the electrostrictive force is:

$$\vec{F} = \frac{\varepsilon_o}{2} \vec{\nabla} (\vec{E}^2 \frac{d\varepsilon_r}{d\rho} \rho)$$

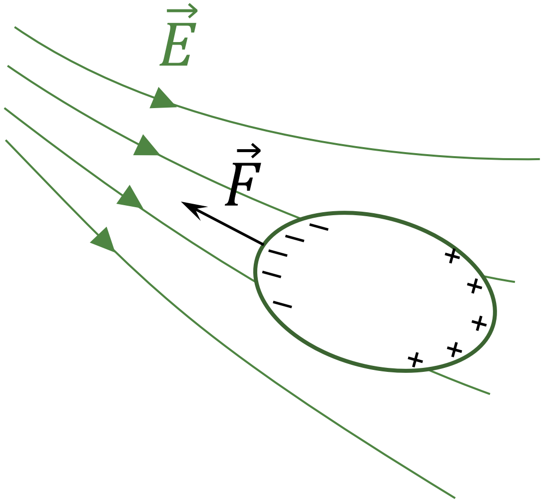
Electrostriction causes a material compression from lower to higher field strengths.
In optics the wave vibration is so fast that this mechanical compression will only feel the time-average field envelope and the squared field can be straightforwardly replaced by the intensity:

$$I = \frac{1}{2} n \varepsilon_o c_o^2 |\vec{E}|^2$$

Replacing by all previously deduced expressions:

$$\vec{F} = \frac{2}{c_o^2} \frac{dn}{d\rho} \rho \vec{\nabla} I = \frac{(n^2 - 1)(n^2 + 2)}{3nc_o^2} \vec{\nabla} I$$

The electrostriction



$$\vec{F}=\frac{2}{c_o^2}\frac{dn}{d\rho}\rho\vec{\nabla}I=\frac{(n^2-1)(n^2+2)}{3nc_o^2}\vec{\nabla}I$$

In nonlinear optics high intensities are required, so that focussed or guided beams are normally presenting a strong intensity gradient:

- ➔ Electrostrictive forces are important in dense media
- ➔ They cause internal pressure and compression
- ➔ Density change causes a refractive index change through elasto-optic effect
- ➔ The net result is an intensity-dependent refractive index $n(I)$

Electrostriction can be seen like the reverse of the elasto-optic effect: $I=\frac{1}{2}n\varepsilon_o c_o^2\left|\vec{E}\right|^2 \quad\Rightarrow\quad \frac{\Delta I}{I}=\frac{\Delta n}{n}$

$$\frac{\Delta I}{I}=\frac{\Delta n}{n}=\frac{\rho}{n}\frac{dn}{d\rho}\frac{\Delta\rho}{\rho}\quad\Rightarrow\quad\frac{\Delta\rho}{\rho}=\frac{n}{\rho}\left(\frac{dn}{d\rho}\right)^{-1}\frac{\Delta I}{I}$$

3rd order nonlinearity

In an *optically isotropic medium* the symmetry prohibits the presence of terms to an even power in the polarization expansion:

$\Rightarrow$ **Nonlinear polarisation**: $\mathscr{P}_{NL} = 4\,\chi^{(3)}\,\mathscr{E}^{3}$

Let consider an incident wave : $\mathscr{E}\left(t\right) = \mathrm{Re}\left\{ E\left(\omega\right) \mathrm{e}^{j\omega(t - n_\omega \frac{z}{c_o})} \right\}$

$$\mathscr{P}_{NL} = 3\,\chi^{(3)} \left|E\left(\omega\right)\right|^{2} \mathrm{Re}\left\{ E\left(\omega\right) \mathrm{e}^{j\omega(t - n_\omega \frac{z}{c_o})} \right\}$$

ω frequency source

$$+\,\chi^{(3)}\,\mathrm{Re}\left\{ E^{3}\left(\omega\right) \mathrm{e}^{j\left(3\omega\right)(t - n_\omega \frac{z}{c_o})} \right\}$$

3ω frequency source

The *3rd harmonic generation* is difficult to achieve, since the source term must propagate at the *same velocity* as the field term ➔ $n_\omega = n_{3\omega}$

3rd order nonlinearity

The ω frequency term will induce a change $\Delta\chi$ in the medium susceptibility:

$$\varepsilon_o\,\Delta\chi = \frac{\mathscr{P}_{NL}(\omega)}{\mathscr{E}(\omega)} = 3\,\chi^{(3)}\left|E(\omega)\right|^2 = 6\,\frac{\chi^{(3)}}{n\,\varepsilon_o\,c_o}\,I$$

using the intensity $I = \frac{n\,\varepsilon_o\,c_o}{2}\left|E(\omega)\right|^2$

that will cause the following change Δn in the refractive index :

$$\Delta n = \frac{\partial n}{\partial\chi}\,\Delta\chi = \frac{\Delta\chi}{2n} = \frac{3}{\varepsilon_o^2\,c_o^2\,n^2}\,\chi^{(3)}\,I = n_2\,I$$

⇒ **Optical Kerr effect**: $\boxed{n(I) = n + n_2\,I}$ with $n_2 = \frac{3\chi^{(3)}}{\varepsilon_o^2\,c_o^2\,n^2}$

Typical value: $n_2 = 3.2\ 10^{-20}$ m^2/W in silica optical fibres.

Nonlinear coefficient

- The phase shift induced by the nonlinear refractive index can be expressed as:

$$\phi_{NL} = k_o\, n_2\, I_o\, L_{eff} = \frac{2\pi\omega}{c_o} n_2 \frac{P_o}{A_{eff}} L_{eff}$$

- A nonlinear coefficient γ is defined, such as:

$$\phi_{NL} = \gamma\, P_o\, L_{eff}\,, \quad \text{so that} \quad \boxed{\gamma = \frac{2\pi\omega\, n_2}{c_o A_{eff}}}$$

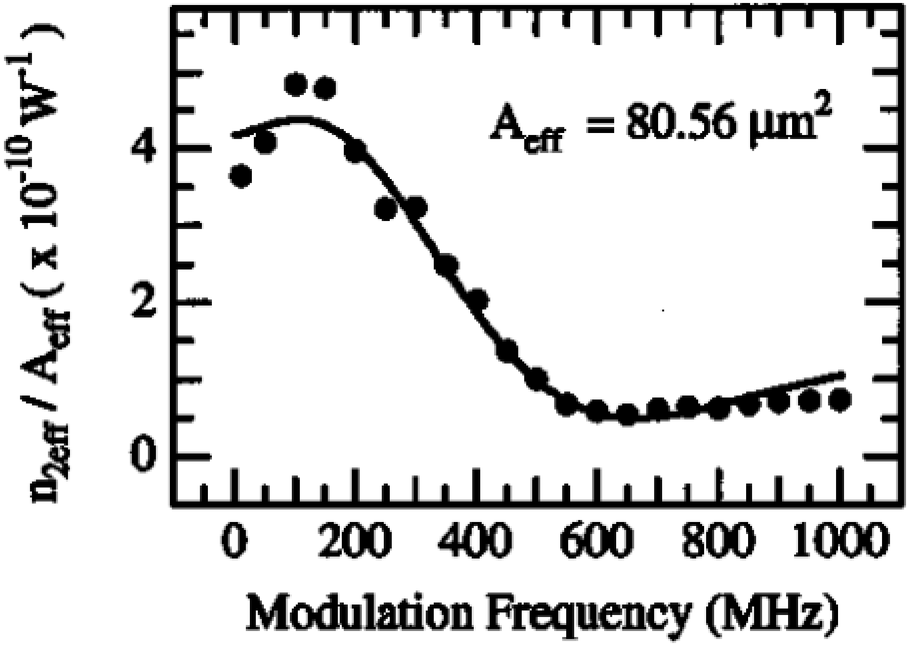
- A typical value in standard silica fibres is $\gamma \simeq 1$ W^{-1} km^{-1}
- A nonlinear length can be defined, such as:

$$\phi_{NL} = 1 \qquad \Rightarrow \qquad \boxed{L_{NL} = \frac{1}{\gamma P_o}}$$

Physical origin of optical Kerr effect

The optical Kerr effect in amorphous dielectric materials (silica, soft glasses, etc…) has essentially 2 origins:

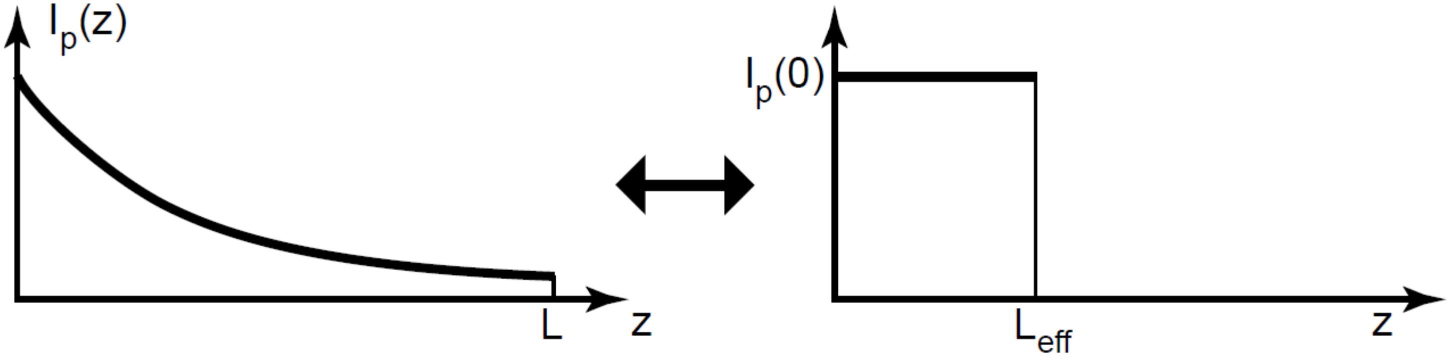
1. The nonlinear response of the induced molecular dipoles in the material to the applied electric field, known as the **electronic contribution**.
   It is weak, but is always present and has no observed frequency limit (fast response)
2. The refractive index change due to the electrostriction induced by a gradient of intensity, known as the **electrostrictive contribution**.
   It is the dominant contribution (~1.5X larger), but is observed only in focussed beams and has a frequency limit of approx. 300 MHz in standard optical fibres (slow response).



Eric L. Buckland and Robert W. Boyd, Opt. Lett. **22**, 676-678 (1997)

Nonlinear effective length

- Since the 3rd order nonlinearity is scaled by the intensity I and the intensity decays during the propagation as a result of the *linear loss* α, it is possible to define a fictitious effective length L_{eff} that would result in the *same nonlinear transformation* for a *lossless* medium



Let's take a general expression for a 3rd order nonlinear interaction, in which an undepleted pump intensity $I_p(z)$ transforms a signal field $E_s(z)$ through a general nonlinear coefficient γ in a medium showing a linear loss α :

$$dE_s = \gamma I_p(z) E_s(z) dz - \frac{\alpha}{2} E_s(z) dz \quad \Rightarrow \quad \frac{dE_s}{E_s} = [\gamma I_p(0) \mathrm{e}^{-\alpha z} - \frac{\alpha}{2}] dz$$

Nonlinear effective length

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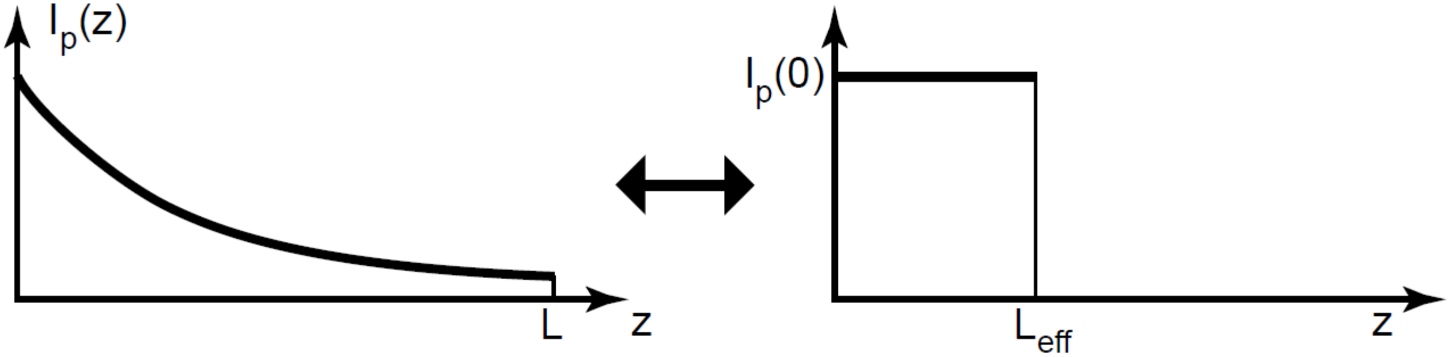


Integrating over *z* from 0 to *L*: $\ln\left[\frac{E_s(L)}{E_s(0)}\right] = \gamma \underbrace{\int_0^L I_p(0)\,\mathrm{e}^{-\alpha z}\,dz}_{I_p(0)L_{eff}} - \frac{\alpha}{2}L$

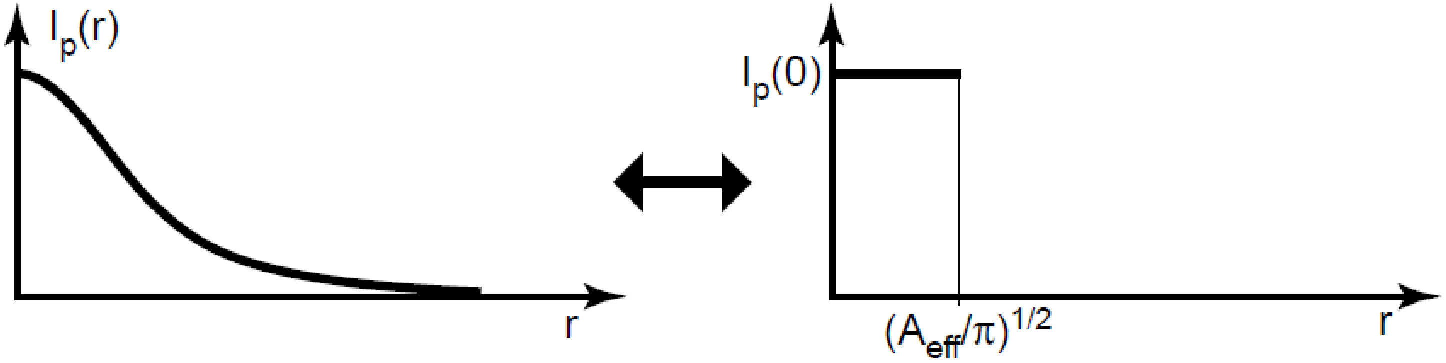
$$\int_0^L I_p(0)\,\mathrm{e}^{-\alpha z}\,dz = I_p(0)\frac{1}{\alpha}(1-\mathrm{e}^{-\alpha L}) \rightarrow \boxed{L_{eff} = \frac{1}{\alpha}(1-\mathrm{e}^{-\alpha L})}$$

$$L_{eff} \overset{L\to\infty}{=} \frac{1}{\alpha} \simeq 22\,\mathrm{km}$$

in silica fibre ($\alpha \le 0.2$ dB/km)

Nonlinear effective area

- In any guided mode or real light beam the intensity *I* shows a transversal distribution *I*(*x*,*y*) that is non-uniform. It is possible to define a fictitious effective area A_{eff} that would result in the *same nonlinear transformation* for an *uniform* intensity distribution.



$$dE_s = \gamma I_p(z) E_s(z) dz - \frac{\alpha}{2} E_s(z) dz$$

Assumption: same transverse field distribution *F*(*x*,*y*) for pump and signal

$$E_s(x,y) = E_s^o\, F(x,y) \qquad E_p(x,y) = E_p^o\, F(x,y)$$

$$I_p(x,y) = I_p^o \left|F(x,y)\right|^2$$

Equation in intensity: $dI_s = d(E_s^* E_s) = E_s^* dE_s + E_s dE_s^* = (\gamma + \gamma^*) I_p I_s dz - \alpha I_s dz$

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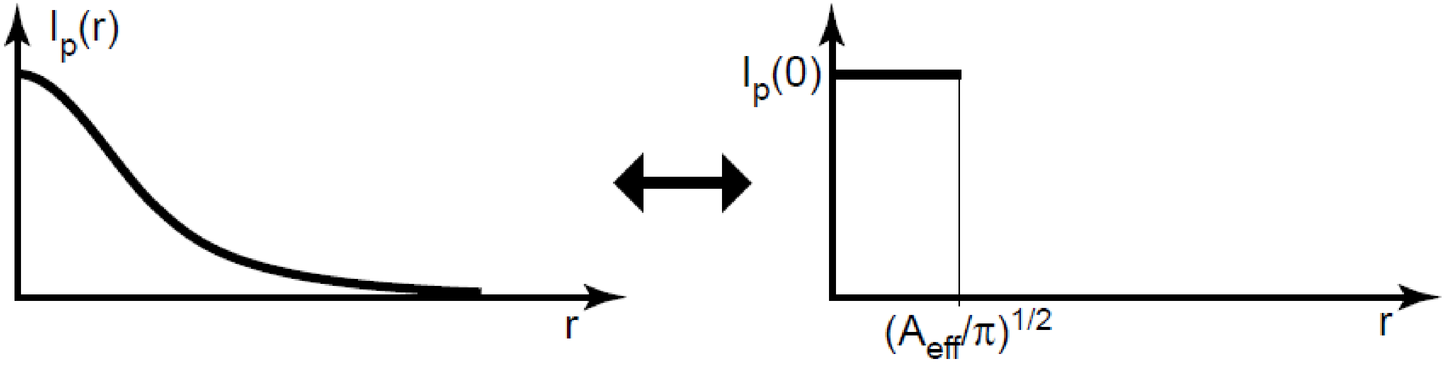

$$\iint dI_s^o \left|F(x,y)\right|^2 dx\,dy = (\gamma+\gamma^*)dz \iint I_p^o \left|F(x,y)\right|^2 I_s^o \left|F(x,y)\right|^2 dx\,dy$$

$$\Rightarrow dI_s^o = (\gamma+\gamma^*) I_p^o I_s^o dz \frac{\iint \left|F(x,y)\right|^4 dx\,dy}{\iint \left|F(x,y)\right|^2 dx\,dy}$$

Effective intensity: $I_{s,p}^{eff} = \frac{P_{s,p}}{A_{eff}} = \frac{I_{s,p}^o \iint \left|F(x,y)\right|^2 dx\,dy}{A_{eff}}$ and, by definition, $dI_s^{eff} = (\gamma+\gamma^*) I_p^{eff} I_s^{eff} dz$

Nonlinear effective area

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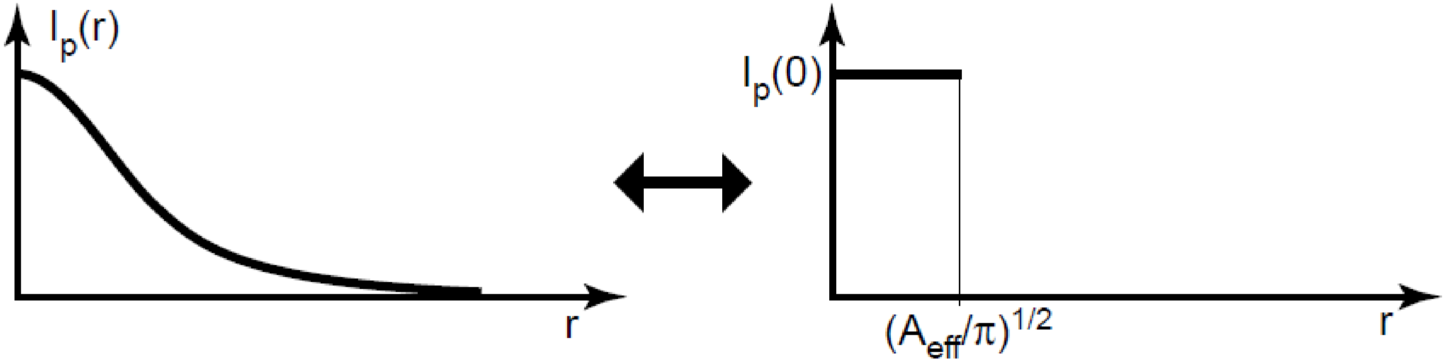

Developing the equation for the effective intensities: $dI_s^o = (\gamma + \gamma^*) I_p^o I_s^o dz \frac{\iint |F(x,y)|^2 \, dx\, dy}{A_{eff}}$

Equating the 2 expressions:

$$\boxed{A_{eff} = \frac{\left(\int\limits_{-\infty}^{+\infty} \int |F(x,y)|^2 \, dx\, dy \right)^2}{\int\limits_{-\infty}^{+\infty} \int |F(x,y)|^4 \, dx\, dy} = \frac{2\pi \left(\int\limits_{-\infty}^{+\infty} |F(r)|^2 \, r\, dr \right)^2}{\int\limits_{-\infty}^{+\infty} |F(r)|^4 \, r\, dr}}$$

$A_{eff} \simeq 80\ \mu\text{m}^2$ in standard single mode fibre (MFD$\sim$10 μm)